International Journal of Applied Mathematical Sciences,
ISSN 0973-0176 Vol.2 No.1(2005), pp. 1-10

http://www.gbspublisher.com/ijams.htm

On Hilbert's Integral Inequality with some parameters

Bicheng Yang
Department of Mathematics,
Guangdong Education Collage, Guangzhou,
Guangdong 510303, P.R. CHINA
E-mail: bcyang@pub.guangzhou.gd.cn

Abstract

In this paper, by introducing some parameters and estimating the weight function, we give a generalization of Hilbert's integral inequality with a best constant factor, which involves the β function. As applications, the equivalent form and some particular results are considered.

2000 Mathematics Subject Classification code: 26D15.

Key Words and phrases: Hilbert's integral inequality, β function, Hölder's inequality.

1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (cf. Hardy et al [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's integral inequality, which is important in analysis and its applications (cf. Mitrinovic et al [2]). And the equivalent form is

$$\int_0^\infty (\int_0^\infty \frac{f(x)}{x+y}dx)^2 dy < \pi^2 \int_0^\infty f^2(x)dx, \tag{1.2}$$

where the constant factor π^2 is still the best possible.

In 1925, Hardy-Riesz gave some classical extended results on (1.1) and (1.2), by introducing (p, q)-parameter as follows (see [3]): If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, f, g are non-negative real functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \{ \int_0^\infty f^p(x)dx\}^{\frac{1}{p}} \{ \int_0^\infty g^q(x)dx\}^{\frac{1}{q}}; \qquad (1.3)$$

$$\int_0^\infty (\int_0^\infty \frac{f(x)}{x+y} dx)^p dy < [\frac{\pi}{\sin(\frac{\pi}{p})}]^p \int_0^\infty f^p(x)dx, \qquad (1.4)$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $[\frac{\pi}{\sin(\pi/p)}]^p$ are all the best possible. Inequality (1.3) is named Hardy-Hilbert's integral inequality, which is equivalent to (1.4). For p=q=2, inequality (1.3) reduces to (1.1), and (1.4) reduces to (1.2).

In 1998, by introducing a parameter $\lambda \in (0, 1]$, and the β function, Yang [4] gave a generalization of (1.1) as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{2}, \frac{\lambda}{2}) \{ \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty x^{1-\lambda} g^2(x)dx\}^{\frac{1}{2}}, \quad (1.5)$$

where B(u,v) is the β function, and (cf. Wang et al [5])

$$B(u, v) := \int_0^\infty \frac{1}{(1+x)^{u+v}} x^{-1+u} dx = B(v, u) \ (u, v > 0). \qquad (1.6)$$

Yang [6, 7, 8] also gave some generalizations of (1.3) and (1.4) as: If $\lambda > 2 - min\{p, q\}$, f, g are non-negative functions such that $0 < \int_0^\infty x^{1-\lambda} f^p(x)dx < \infty \ and \ 0 < \int_0^\infty x^{1-\lambda} g^q(x)dx < \infty$, then we have the following two equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$$
$$\times \{ \int_0^\infty x^{1-\lambda} f^p(x)dx\}^{1/p} \{ \int_0^\infty x^{1-\lambda} g^q(x)dx\}^{1/q}; \qquad (1.7)$$

$$\int_0^\infty y^{(p-1)(\lambda-1)} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy$$
$$< [B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})]^p \int_0^\infty x^{1-\lambda} f^p(x)dx, \qquad (1.8)$$

where the constant factors $B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ and $[B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})]^p$ are all the best possible. For $\lambda = 1$, (1.7) reduces to (1.3), and (1.8) reduces to (1.4); and for p=q=2, (1.7) reduces to (1.5) suited $\lambda > 0$. It follows that (1.7) and (1.8) are generalizations of (1.1) and (1.2) with (p,q)-parameter and λ. Recently, [9, 10] considered some multiple extensions of (1.1) and (1.3), and [11,12,13] gave some extended series form of (1.1)and some new form of (1.1). In 2003, Yang [14] provided an extensive account of the above results.

In this paper, by using the β function and obtaining the expression of the weight function, we give a new generalization of (1.1) with some parameters , such that both (1.5) and (1.7) are its particular results. As applications, we also consider the equivalent form and some other new particular results.

2. Some Lemmas

Lemma 2.1. If $t \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{s} = 1$, and $\lambda > (2 - min\{r, s\})t$, define the weight function $\omega_{\lambda,t}(s, p, x)$ as

$$\omega_{\lambda,t}(s,p,x) := \int_0^\infty \frac{1}{(x+y)^\lambda} \cdot \frac{x^{p[(1-t)r+2t-\lambda]/(qr)}}{y^{[(1-t)s+2t-\lambda]/s}} dy. \tag{2.1}$$

The we have

$$\omega_{\lambda,t}(s,p,x) = B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}) x^{p[1-t+\frac{2t-\lambda}{r}]-1}. \tag{2.2}$$

Proof. Setting u=y/x in the integral of (2.1), by (1.5) we find

$$\begin{aligned} \omega_{\lambda,t}(s,p,x) &= x^{-1+p[1-t+\frac{2t-\lambda}{r}]} \int_0^\infty \frac{1}{(1+u)^\lambda} u^{-1+[(s-2)t+\lambda]/s} du \\ &= x^{p[1-t+\frac{2t-\lambda}{r}]-1} B(\frac{(s-2)t+\lambda}{s}, \frac{(r-2)t+\lambda}{r}). \end{aligned} \tag{2.3}$$

Hence, (2.2) is valid and the lemma is proved.

Note. By (2.3), we still have

$$\begin{aligned} \omega_{\lambda,t}(r,q,y) &= \int_0^\infty \frac{1}{(x+y)^\lambda} \cdot \frac{y^{q[(1-t)s+2t-\lambda]/(ps)}}{x^{[(1-t)r+2t-\lambda]/r}} dx \\ &= B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}) y^{q[1-t+\frac{2t-\lambda}{s}]-1}. \end{aligned} \tag{2.4}$$

Lemma 2.2. On the assumption of Lemma 2.1, if $\varepsilon > 0$ is small enough $(\varepsilon < \frac{p}{r}[(r-2)t+\lambda])$, then we have

$$\begin{aligned} I &:= \int_1^\infty \{\int_1^\infty \frac{x^{t-1-\frac{\varepsilon}{p}+\frac{\lambda-2t}{r}}}{(x+y)^\lambda} dx\} y^{t-1-\frac{\varepsilon}{q}+\frac{\lambda-2t}{s}} dy \\ &\geq \frac{1}{\varepsilon} B(\frac{(r-2)t+\lambda}{r} - \frac{\varepsilon}{p}, \frac{(s-2)t+\lambda}{s} + \frac{\varepsilon}{p}) - O(1) \ (\varepsilon \to 0^+). \end{aligned} \tag{2.5}$$

Proof. Setting u=x/y in expression I, by (1.6) we find

$$\begin{aligned}
I &= \int_1^\infty y^{-1-\varepsilon}[\int_{1/y}^\infty \frac{1}{(1+u)^\lambda} u^{-1+\frac{(r-2)t+\lambda}{r}-\frac{\varepsilon}{p}} du]dy \\
&= \int_1^\infty y^{-1-\varepsilon}[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{-1+\frac{(r-2)t+\lambda}{r}-\frac{\varepsilon}{p}} du]dy \\
&\quad - \int_1^\infty y^{-1-\varepsilon}[\int_0^{1/y} \frac{1}{(1+u)^\lambda} u^{-1+\frac{(r-2)t+\lambda}{r}-\frac{\varepsilon}{p}} du]dy \\
&\geq \frac{1}{\varepsilon} B(\frac{(r-2)t+\lambda}{r} - \frac{\varepsilon}{p}, \frac{(s-2)t+\lambda}{s} + \frac{\varepsilon}{p}) \\
&\quad - \int_1^\infty y^{-1}[\int_0^{1/y} u^{-1+\frac{(r-2)t+\lambda}{r}-\frac{\varepsilon}{p}} du]dy \\
&= \frac{1}{\varepsilon} B(\frac{(r-2)t+\lambda}{r} - \frac{\varepsilon}{p}, \frac{(s-2)t+\lambda}{s} + \frac{\varepsilon}{p}) - [\frac{(r-2)t+\lambda}{r} - \frac{\varepsilon}{p}]^{-2}.
\end{aligned}$$

Hence, (2.5) is valid, and the lemma is proved.

3. Main Results and Applications

Theorem 3.1. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{s} = 1, t \geq 0,$ and $\lambda > (2 - min\{r, s\})t,$
f, g are non-negative real functions, such that

$$0 < \int_0^\infty x^{p(1-t+\frac{2t-\lambda}{r})-1} f^p(x)dx < \infty \ and \ 0 < \int_0^\infty x^{q(1-t+\frac{2t-\lambda}{s})-1} g^q(x)dx < \infty,$$

then we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}) \\
&\times\{\int_0^\infty x^{p(1-t+\frac{2t-\lambda}{r})-1} f^p(x)dx\}^{1/p}\{\int_0^\infty x^{q(1-t+\frac{2t-\lambda}{s})-1} g^q(x)dx\}^{1/q}, \qquad (3.1)
\end{aligned}$$

where the constant factor $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$ is the best possible. In particular, for r=s=2, we have $\lambda > 0$, and

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{2}, \frac{\lambda}{2}) \\
&\times\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}}. \qquad (3.2)
\end{aligned}$$

Proof. By Hölder's inequality, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy = \int_0^\infty\int_0^\infty [\frac{f(x)}{(x+y)^{\lambda/p}}$$

$$\times\frac{x^{[(1-t)r+2t-\lambda]/(qr)}}{y^{[(1-t)s+2t-\lambda]/(ps)}}][\frac{g(y)}{(x+y)^{\lambda/q}}\cdot\frac{y^{[(1-t)s+2t-\lambda]/(ps)}}{x^{[(1-t)r+2t-\lambda]/(qr)}}]dxdy$$

$$\leq\{\int_0^\infty[\int_0^\infty \frac{1}{(x+y)^\lambda}\cdot\frac{x^{p[(1-t)r+2t-\lambda]/(qr)}}{y^{[(1-t)s+2t-\lambda]/s}}dy]f^p(x)dx\}^{1/p}$$

$$\times\{\int_0^\infty[\int_0^\infty \frac{1}{(x+y)^\lambda}\cdot\frac{y^{q[(1-t)s+2t-\lambda]/(ps)}}{x^{[(1-t)r+2t-\lambda]/r}}dx]g^q(y)dy\}^{1/q}. \tag{3.3}$$

If (3.3) takes the form of equality, then there exists constants A and B, such they are not all zero that (see [15])

$$A\frac{f^p(x)}{(x+y)^\lambda}\cdot\frac{x^{p[(1-t)r+2t-\lambda]/(qr)}}{y^{[(1-t)s+2t-\lambda]/s}}$$

$$=B\frac{g^q(y)}{(x+y)^\lambda}\cdot\frac{y^{q[(1-t)s+2t-\lambda]/(ps)}}{x^{[(1-t)r+2t-\lambda]/r}},\ a.e.\ in\ (0,\infty)\times(0,\infty).$$

We find that

$$Ax^{p[1-t+\frac{2t-\lambda}{r}]}f^p(x) = By^{q[1-t+\frac{2t-\lambda}{s}]}g^q(y)\ a.e.\ in\ (0,\infty)\times(0,\infty).$$

Hence there exists a constant C, such that

$$Ax^{p[1-t+\frac{2t-\lambda}{r}]}f^p(x) = C = By^{q[1-t+\frac{2t-\lambda}{s}]}g^q(y),\ a.e.\ in\ (0,\infty).$$

Not lose generality, suppose that $A\neq 0$, we may get

$$x^{p[1-t+\frac{2t-\lambda}{r}]-1}f^p(x) = C/(Ax),\ a.e.\ in\ (0,\infty),$$

which contradicts the fact that $0<\int_0^\infty x^{p[1-t+\frac{2t-\lambda}{r}]-1}f^p(x)dx<\infty$. Hence, by (2.1), we can rewrite (3.3) as

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < \{\int_0^\infty \omega_{\lambda,t}(s,p,x)f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty \omega_{\lambda,t}(r,q,y)g^q(y)dy\}^{\frac{1}{q}}, \tag{3.4}$$

and by (2.2) and (2.4), we have (3.1).

For $\varepsilon>0$ small enough $(\varepsilon<\frac{p}{r}[(r-2)t+\lambda])$, setting $f_\varepsilon, g_\varepsilon$ as:

$$f_\varepsilon(x) = g_\varepsilon(x) = 0,\ for\ x\in(0,1);$$

$$f_\varepsilon(x) = x^{t-1-\frac{\varepsilon}{p}+\frac{\lambda-2t}{r}},\ g_\varepsilon(x) = x^{t-1-\frac{\varepsilon}{q}+\frac{\lambda-2t}{s}},\ for\ x\in[1,\infty),$$

then we find

$$J := \{\int_0^\infty x^{p[1-t+\frac{2t-\lambda}{r}]-1} f_\varepsilon^p(x)dx\}^{\frac{1}{p}} \{\int_0^\infty x^{q[1-t+\frac{2t-\lambda}{s}]-1} g_\varepsilon^q(x)dx\}^{\frac{1}{q}} = \frac{1}{\varepsilon}.$$

If the constant factor $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$ in (3.1) is not the best possible, then there exists a positive constant K (with $K < B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$), such (3.1) is still valid that we replace $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$ by K. In particular, by (2.5), we have

$$B(\frac{(r-2)t+\lambda}{r} - \frac{\varepsilon}{p}, \frac{(s-2)t+\lambda}{s} + \frac{\varepsilon}{p}) - \varepsilon O(1)$$

$$\leq \varepsilon I = \varepsilon \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x) g_\varepsilon(y)}{(x+y)^\lambda} dxdy < \varepsilon K J = K.$$

For $\varepsilon \to 0^+$, it follows that $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}) \leq K$, which contradicts the fact that $K < B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$. Hence the constant factor $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$ in (3.1) is the best possible. The theorem is proved.

Theorem 3.2. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{s} = 1, t \geq 0,$ and
$\lambda > (2 - min\{r, s\})t,$
f is a non-negative real function, and $0 < \int_0^\infty x^{p(1-t+\frac{2t-\lambda}{r})-1} f^p(x)dx < \infty$, then we have

$$\int_0^\infty y^{\frac{p}{s}[(s-2)t+\lambda]-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy$$
$$< [B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})]^p \int_0^\infty x^{p(1-t+\frac{2t-\lambda}{r})-1} f^p(x)dx, \qquad (3.5)$$

where the constant factor $[B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})]^p$ is the best possible; Inequality (3.5) is equivalent to (3.1). In particular, for r=s=2, we have $\lambda > 0$, and

$$\int_0^\infty y^{\frac{p\lambda}{2}-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy < [B(\frac{\lambda}{2}, \frac{\lambda}{2})]^p \int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx. \qquad (3.6)$$

Proof. Setting a real function g(y) as

$$g(y) := y^{\frac{p[(s-2)t+\lambda]}{s}-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^{p-1}, \ y \in (0, \infty),$$

then by (3.1), we find

$$\int_0^\infty y^{q(1-t+\frac{2t-\lambda}{s})-1} g^q(y)dy = \int_0^\infty y^{\frac{p}{s}[(s-2)t+\lambda]-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \le B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$$
$$\times \{ \int_0^\infty x^{p(1-t+\frac{2t-\lambda}{r})-1} f^p(x)dx\}^{1/p} \{ \int_0^\infty x^{q(1-t+\frac{2t-\lambda}{s})-1} g^q(x)dx\}^{1/q}. \tag{3.7}$$

Hence we obtain

$$0 < \{ \int_0^\infty y^{q(1-t+\frac{2t-\lambda}{s})-1} g^q(y)dy\}^{1/p} \le B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$$
$$\times \{ \int_0^\infty x^{p(1-t+\frac{2t-\lambda}{r})-1} f^p(x)dx\}^{1/p} < \infty. \tag{3.8}$$

By (3.1), both (3.7) and (3.8) take the form of strict inequality, and we have (3.5).

On the other hand , suppose that (3.5) is valid. By Hölder's inequality, we find

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy = \int_0^\infty [y^{\frac{(s-2)t+\lambda}{s} - \frac{1}{p}} \int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx][y^{-\frac{(s-2)t+\lambda}{s} + \frac{1}{p}} g(y)]dy$$

$$\le \{ \int_0^\infty y^{\frac{p}{s}[(s-2)t+\lambda]-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy\}^{\frac{1}{p}} \{ \int_0^\infty y^{q(1-t+\frac{2t-\lambda}{s})-1} g^q(y)dy\}^{\frac{1}{q}}. \tag{3.9}$$

Then by (3.5), we have (3.1). Hence (3.1) and (3.5) are equivalent.

If the constant factor $[B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})]^p$ in (3.5) is not the best possible, by the same way of showing Theorem 3.1, and using (3.9), we may get a contradiction that the constant factor in (3.1) is not the best possible. Thus we complete the proof of the theorem.

Remark 3.3. (a) For $\lambda = 1$, and p=q=2, (3.2) and (3.6) reduce respectively to (1.1) and (1.2). It follows that (3.1) is a generalization of (1.2) with some parameters and (3.5) is its equivalent form, which is a generalization of (1.2).

(b) For r=p , s=q , and t=1, (3.1) and (3.5) reduce respectively to (1.7) and (1.8). It follows that (3.1) and (3.5) are generalizations of (1.7) and (1.8).

For r=p, s=q and t=0, by reducing (3.1) and (3.5),we have

Corollary 3.4. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $\lambda > 0$, f, g are non-negative real functions, and

$$0 < \int_0^\infty x^{p-\lambda-1} f^p(x)dx < \infty \, and \; 0 < \int_0^\infty x^{q-\lambda-1} g^q(x)dx < \infty,$$

then we have the following two equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{p}, \frac{\lambda}{q}) \{ \int_0^\infty x^{p-\lambda-1} f^p(x)dx\}^{\frac{1}{p}} \{ \int_0^\infty x^{q-\lambda-1} g^q(x)dx\}^{\frac{1}{q}}; \tag{3.10}$$

$$\int_0^\infty y^{(p-1)\lambda-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy < [B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p \int_0^\infty x^{p-\lambda-1} f^p(x)dx, \tag{3.11}$$

where the constant factors $B(\frac{\lambda}{p}, \frac{\lambda}{q})$ and $[B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p$ are all the best possible.

For r=q, s=p and t=0, by (3.1) and (3.5), we have

Corollary 3.5. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $\lambda > 0$, f, g are non-negative real functions, such that

$$0 < \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx < \infty \ and \ 0 < \int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)dx < \infty,$$

then we have the following two equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{\lambda}{p}, \frac{\lambda}{q}) \times \{\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx\}^{1/p} \{\int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)dx\}^{1/q}; \tag{3.12}$$

$$\int_0^\infty y^{\lambda-1} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy < [B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx, \tag{3.13}$$

where the constant factors $B(\frac{\lambda}{p}, \frac{\lambda}{q})$ and $[B(\frac{\lambda}{p}, \frac{\lambda}{q})]^p$ are all the best possible.

For r=q, s=p and t=1, by (3.1) and (3.5), we have

Corollary 3.6. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $\lambda > 2 - min\{p, q\}$, f, g are non-negative real functions, such that

$$0 < \int_0^\infty x^{(p-1)(2-\lambda)-1} f^p(x)dx < \infty, \ and \ 0 < \int_0^\infty x^{(q-1)(2-\lambda)-1} g^q(x)dx < \infty,$$

then we have the following two equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}) \times \{\int_0^\infty x^{(p-1)(2-\lambda)-1} f^p(x)dx\}^{1/p} \{\int_0^\infty x^{(q-1)(2-\lambda)-1} g^q(x)dx\}^{1/q}; \tag{3.14}$$

$$\int_0^\infty y^{p+\lambda-3} [\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx]^p dy < [B(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q})]^p \int_0^\infty x^{(p-1)(2-\lambda)-1} f^p(x)dx, \tag{3.15}$$

where the constant factors $B(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q})$ and $[B(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q})]^p$ are all the best possible. In particular, for $\lambda = 1$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \{\int_0^\infty x^{p-2} f^p(x)dx\}^{\frac{1}{p}} \{\int_0^\infty x^{q-2} g^q(x)dx\}^{\frac{1}{q}}; \tag{3.16}$$

$$\int_0^\infty y^{p-2} (\int_0^\infty \frac{f(x)}{x+y} dx)^p dy < [\frac{\pi}{\sin(\pi/p)}]^p \int_0^\infty x^{p-2} f^p(x)dx, \tag{3.17}$$

Remark 3.7. (a) For $\lambda = 1$, both (3.12) and (1.7) reduce to (1.3), and both (3.13) and (1.8) reduce to (1.4), as (3.14) and (3.15), inequalities (3.10) and (3.11) reduce respectively to (3.16) and (3.17). (b) It is interesting that (1.3) and (3.16) are deferent, although both of them are generalizations of (1.1) with (p, q)-parameter and the same best constant factor. (c) It is also interesting that there are more than four kinds of generalization of (1.5) with the best constant factors only related (p, q)-parameter and $\lambda > 0$, such as (1.7), (3.10), (3.12) and (3.14).

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities. Cambridge Univ. Press, London, 1952.

[2] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, Inequalities involving functions and their integrals and derivatives. Kluwer Academic, Boston, 1991.

[3] G. H. Hardy, Not on a theorwm of Hilbert concerning series of positive terms, Proc. London Math. Soc. 23:2(1925), Records of Proc. xlv-xlvi.

[4] Yang Bicheng, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220**(1998), pp. 778-785.

[5] Wang Zhuxi and Guo Dunrin, An introduction to special functions, Science Press, Beijing, 1979.

[6] Yang Bicheng, On a general Hardy-Hilbert's integral inequality with a best value, *Chinese Annals of Math.* **21A**, (4)(2000), pp. 401-408.

[7] Yang Bicheng, On Hardy-Hilbert's integral inequality, *J. Math. Anal. Appl.*, **261** (2001), pp. 295-306.

[8] Yang Bicheng, On Hardy-Hilbert's integral inequality and its equivalent form, *North. Math. J.*, **19**(2)(2003), pp. 139-148.

[9] Hong Yong, All-sided generalization about Hardy-Hilbert integral inequalities, *Acta Mathematica Sinica,* **44**(4)(2001), pp. 619-626.

[10] He Leping, Yu Jingming and Gao Mingzhe, An extension of Hilbert's integral inequality, *J. Shaoguan Univ.*(Natural Science),**23**(3)(2002), pp. 25-30.

[11] Bicheng Yang, On a generalization of Hilbert's double series theorem, *Math. Ineq. Appl.* bf5(2)(2002), pp. 197-204.

[12] W. T. Sulaiman, On Hardy-Hilbert's integral inequality, *Journal of Inequalities in Pure and Applied Mathematics*, **5**(2)(2004), Article 25.

[13] Yang Bicheng, Ilko Brnetic, Mario Krnic and Josip Pecaric, Generalization of Hilbert's and Hardy-Hilbert's integral inequality, **Math. Ineq. Appl.**(to appear).

[14] Bicheng Yang and T. M. Rassias, On the way of weight coefficient and research for the Hilbert-type inequalities, **Math. Ineq. Appl. 6**(4)(2003), pp. 625-658.

[15] Kuang Jichang, *Applied inequalities*, Shangdong Science and Technology Press, Jinan, 2003.

Research supported by Natural Science Foundation of Guangdong Institutions of Higher Learning, College and University (China, No. 0177).